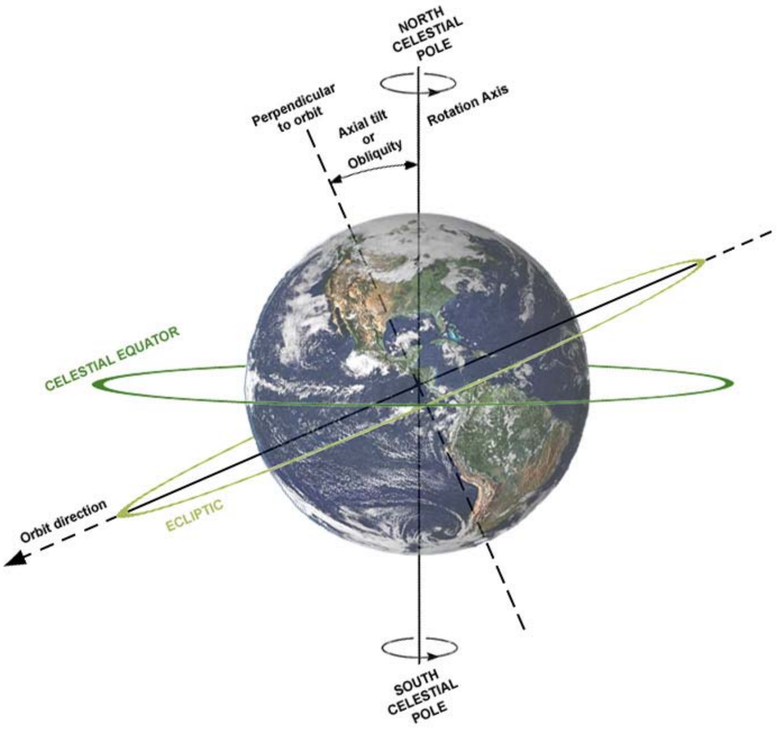
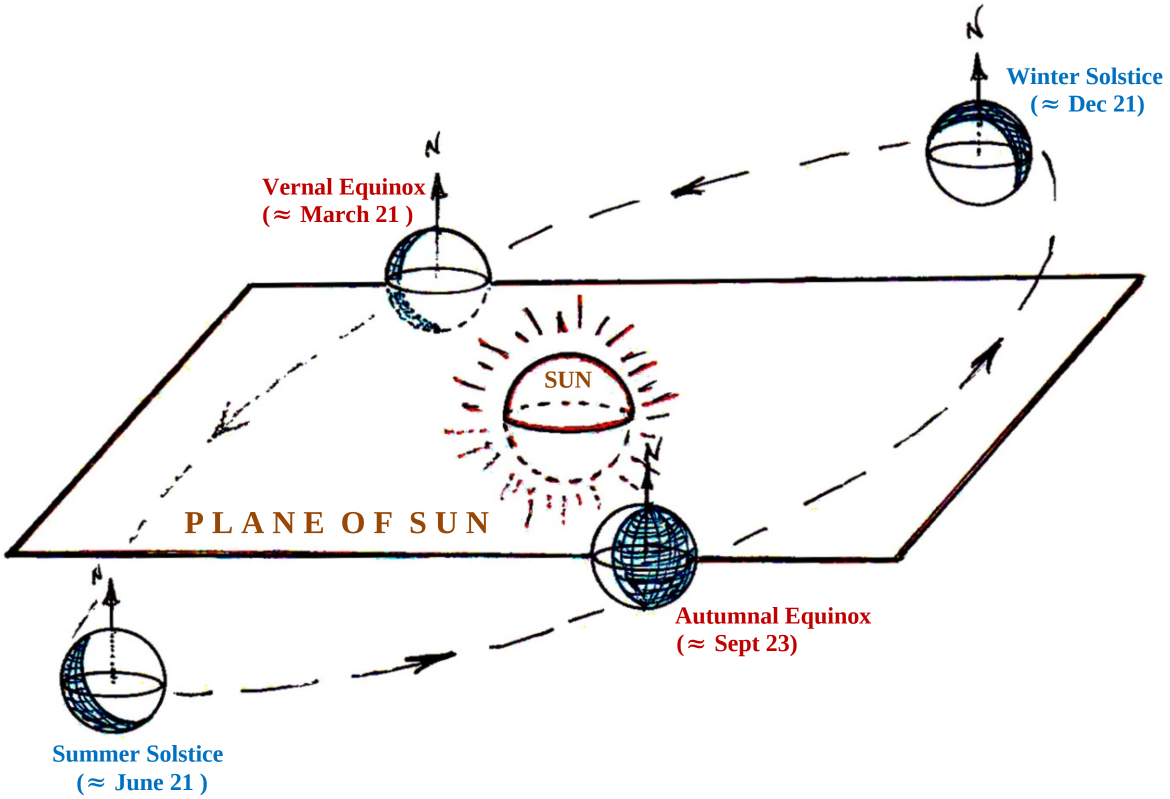
Solar Geometry



The Earth's daily rotation about the axis through its two celestial poles (North and South) is perpendicular to the equator, but it is not perpendicular to the plane of the Earth's orbit. In fact, the measure of tilt or obliquity of the Earth's axis to a line perpendicular to the plane of its orbit is currently about 23.5°.

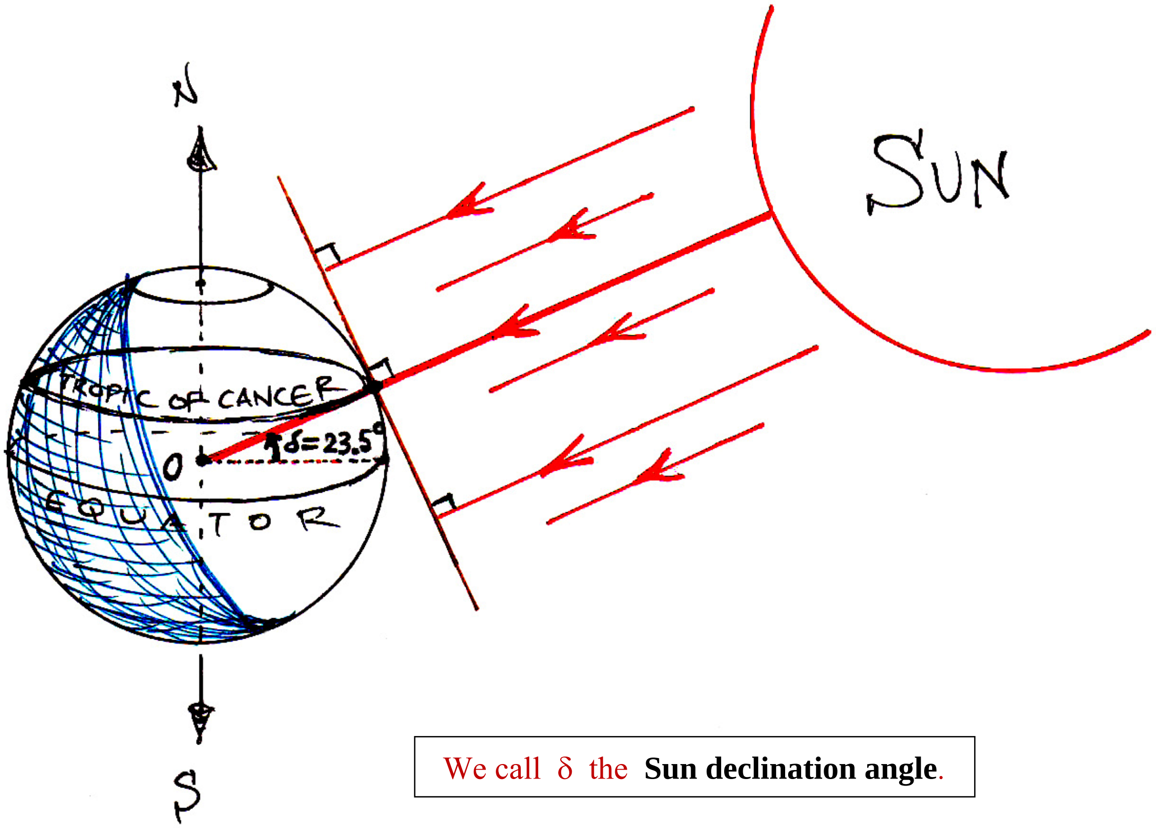
We call the plane parallel to the Earth's celestial equator and through the center of the sun the **plane of the Sun.**

The Earth passes alternately above and below this plane making one complete elliptic cycle every year.



Summer Solstice

On the occasion of the summer solstice, the Sun shines down most directly on the Tropic of Cancer in the northern hemisphere, making an angle $\delta = +23.5°$ with the equatorial plane.



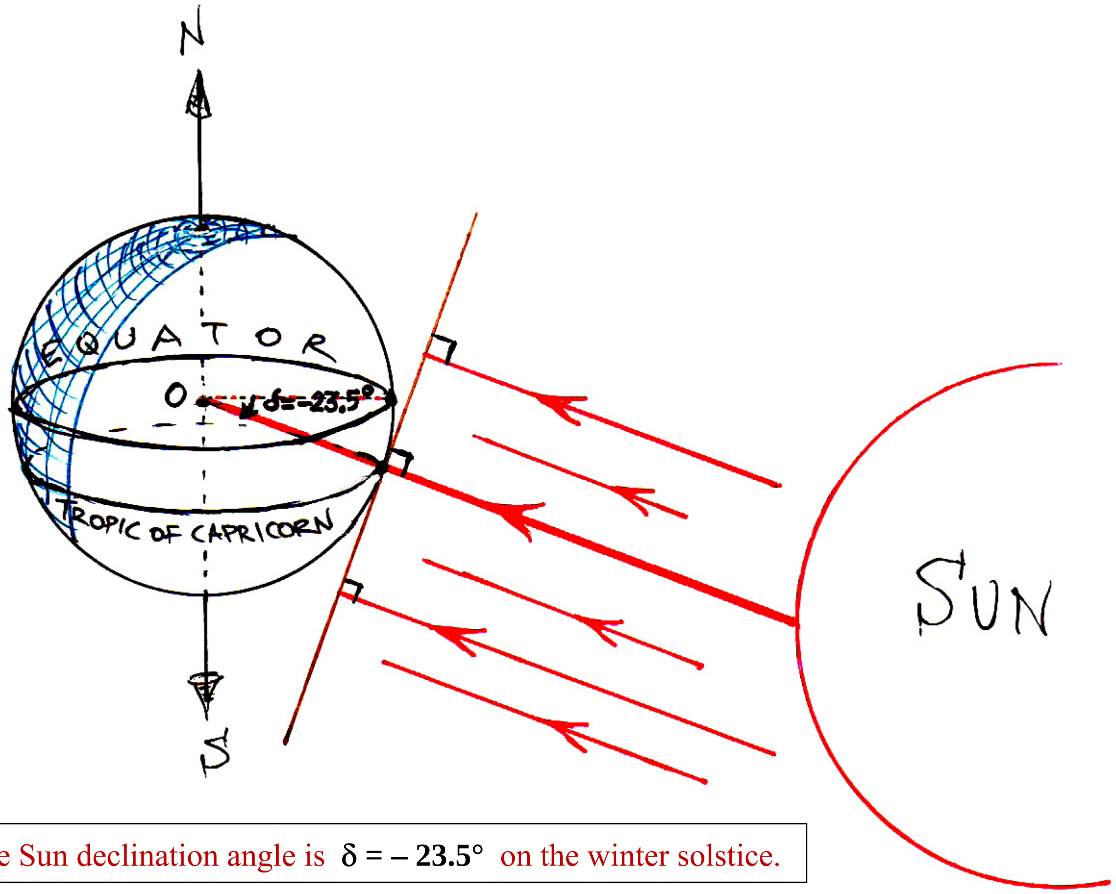
We call δ the **Sun declination angle**.

In general, the **Sun declination angle**, δ, is defined to be that angle made between a ray of the Sun, when extended to the center of the earth, O, and the equatorial plane. We take δ to be positively oriented whenever the Sun's rays reach O by passing through the Northern hemisphere.

On the day of the summer solstice, the sun is above the horizon for the longest period of time in the northern hemisphere. Hence, it is the longest day for daylight there. Conversely, the Sun remains below the horizon at all points within the Antarctic Circle on this day.

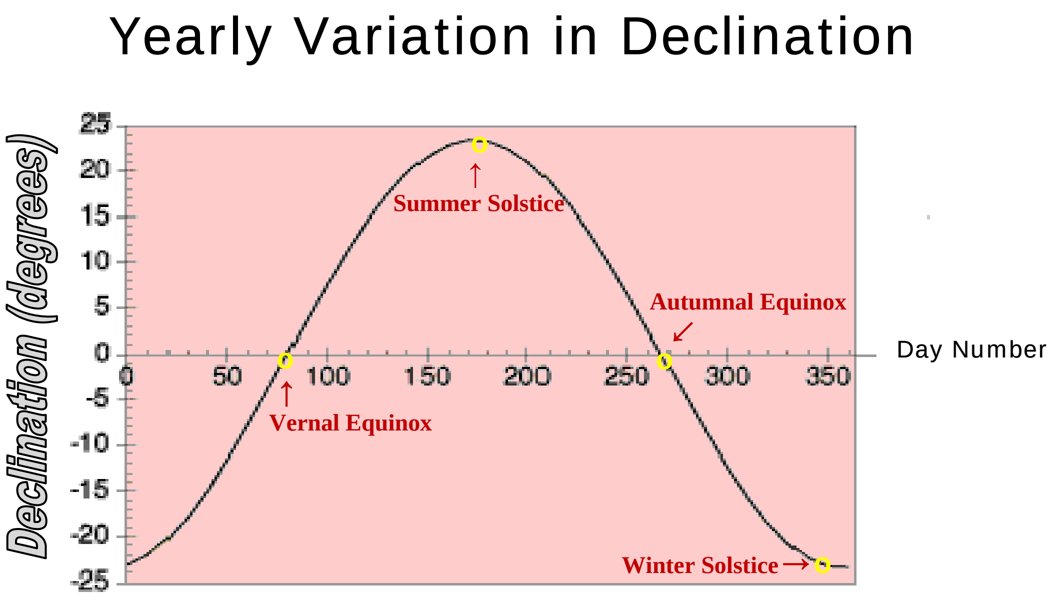
Winter Solstice

On the day of the winter solstice, the smallest portion of the northern hemisphere is exposed to the Sun and the Sun is above the horizon for the shortest period of time there. In fact, the Sun remains below the horizon everywhere within the Arctic Circle on this day. The Sun shines down most directly on the tropic of Capricorn in the southern hemisphere on the occasion of the winter solstice.



The Sun declination angle is **δ = – 23.5°** on the winter solstice.

The Sun declination angle, δ, has the range: $-23.5° \leq \delta \leq +23.5°$ during its yearly cycle .



Accurate knowledge of the declination angle is important in navigation and astronomy. For most solar design purposes, however, an approximation accurate to within about 1 degree is adequate. One such approximation for the declination angle is:

$$\sin\delta = 0.39795\cdot\cos\left[\,0.98563\cdot(N-173)\,\right] \qquad (1)$$

where the argument of the cosine here is in degrees and N denotes the number of days since January 1.

Vernal and Autumnal Equinox

There are two occasions throughout the year when the center of the Earth lies in the plane of the Sun. Since the Earth's North – South axis of rotation is perpendicular to this plane, it follows that on these two days every location on the Earth receives 12 hours of sunshine. These two events are known as the **vernal** and **autumnal equinoxes**.

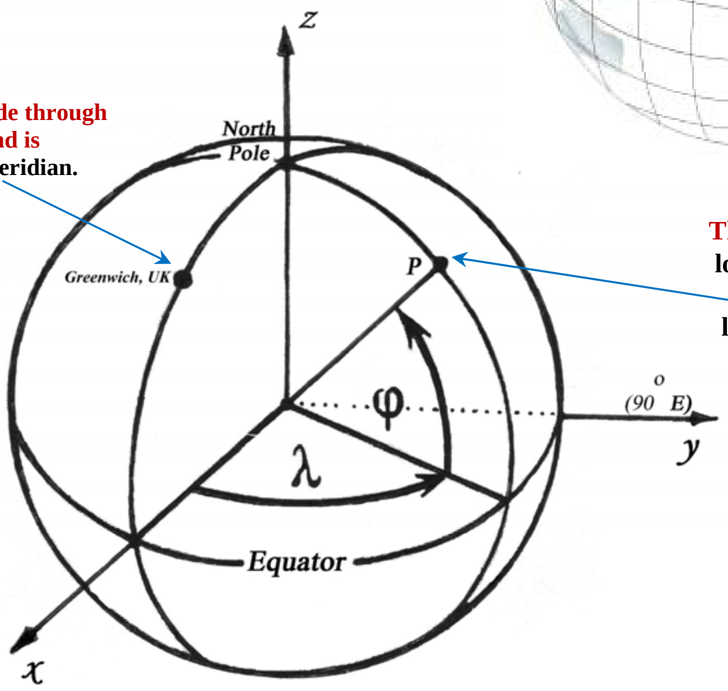
The Sun declination angle has measure: $\delta = 0°$ on the days of the vernal and the autumnal equinox.

Note that

- The Earth is **above** the plane of the Sun during its motion from the autumnal equinox to winter solstice to vernal equinox. Hence, $\delta < 0$ during the fall and winter.
- The Earth is **below** the plane of the sun as it moves from vernal equinox to summer solstice and back to autumnal equinox (i.e. during spring and summer). So $\delta > 0$ during these seasons.

The **latitude** of a location on the Earth is the angle between the line joining that location to the center of the earth and the equatorial plane. For example, Chicago, Illinois is on the circle of latitude 41.8° N. All locations at the same latitude experience the same geometric relationship with the sun.

The great semicircles along the surface of the Earth joining the North to the South poles are called **lines of longitude**.



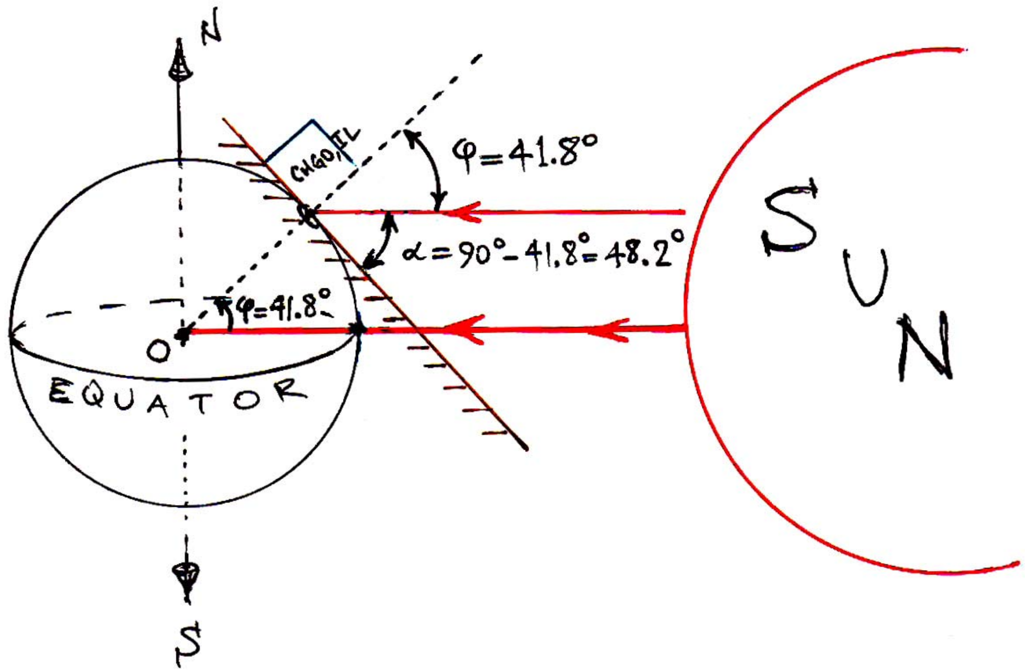
Solar Noon

Solar Noon is defined to be that time of day at which the Sun's rays are directed perpendicular to a given line of longitude. Thus, solar noon occurs at the same instant for all locations along any common line of longitude.

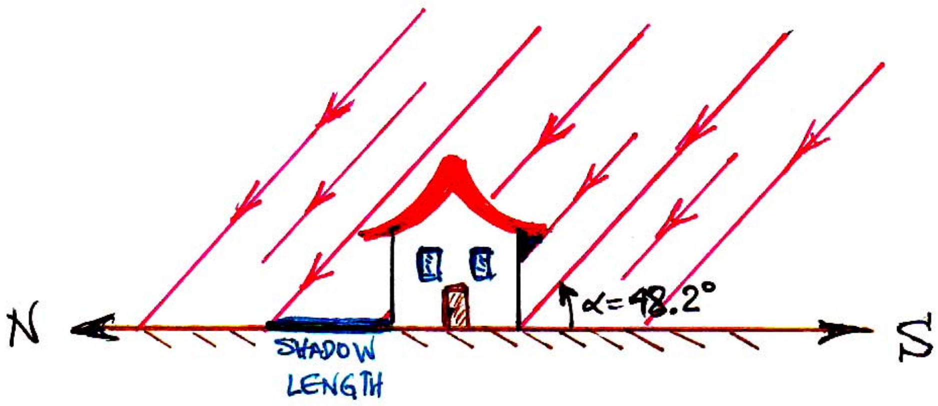
Solar Noon will occur one hour earlier for every 15 degrees of longitude to the east of a given line and one hour later for every 15 degrees west. (This is because it takes the Earth 24 hours to rotate 360°.)

Problem: Determine the elevation angles at which the sun's rays reach the surface of the Earth in Chicago, IL at solar noon at the times of an equinox or a solstice.

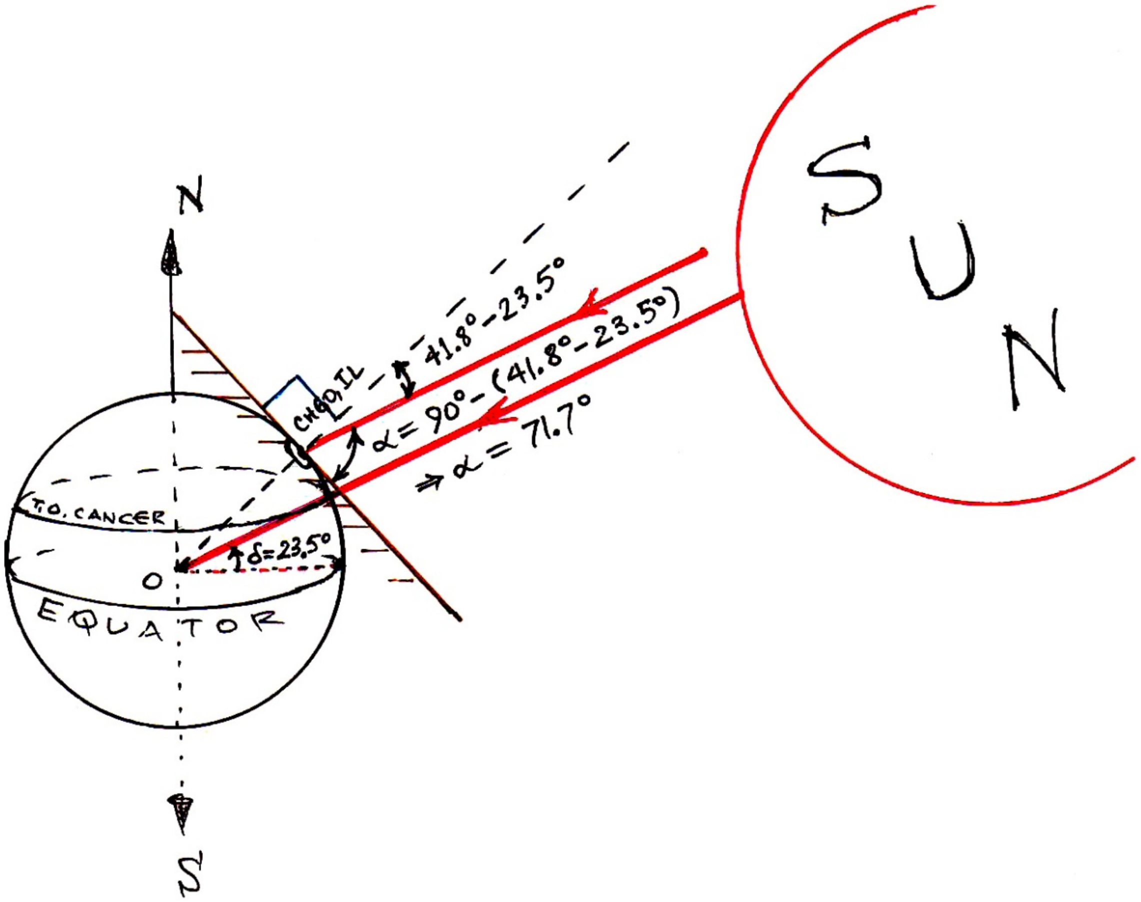
- **At the Vernal and the Autumnal Equinox**



Thus, the most direct angle at which the Sun's rays strike the ground in Chicago is at about 48° on the days of a vernal and autumnal equinox.

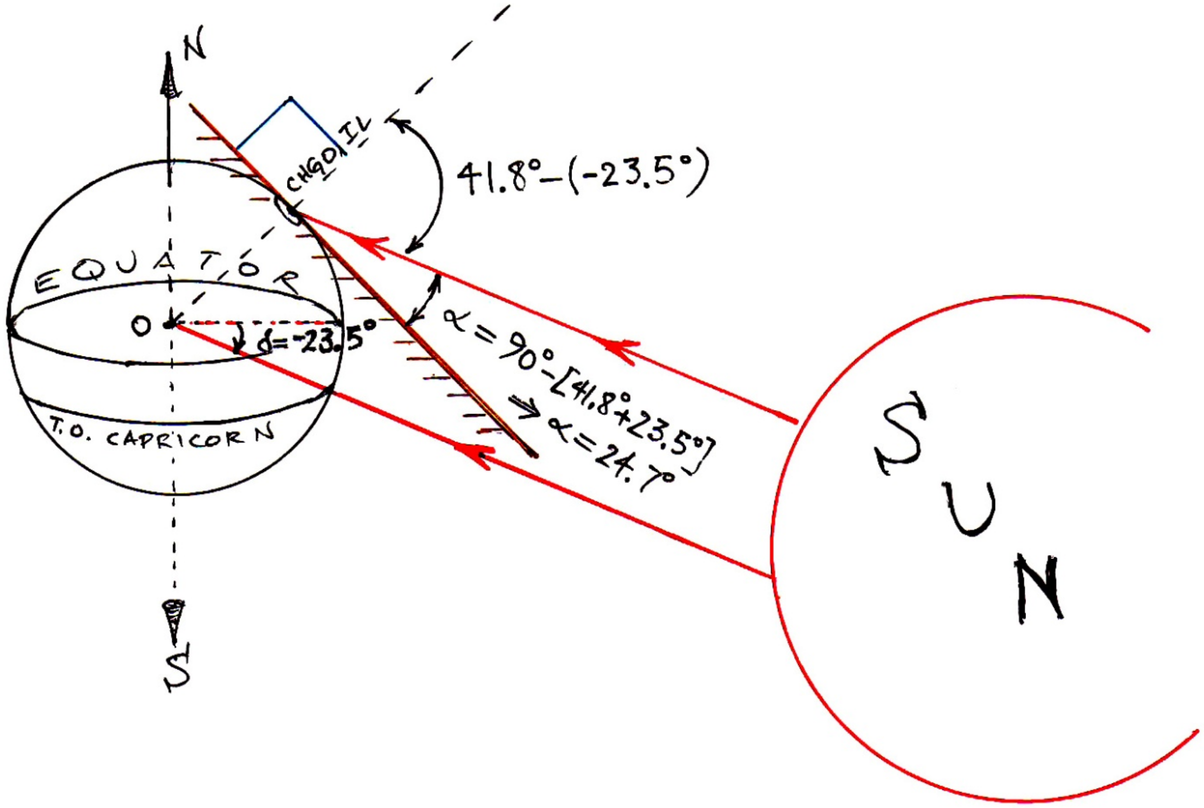


- **At the Summer Solstice**



Thus, the Sun's rays strike the ground in Chicago most directly at a vertical angle of about 72° on the day of the summer solstice.

- **At the Winter Solstice**



On the day of the winter solstice, the Sun's rays strike the ground at an angle of about 25° at solar noon in Chicago, IL.

It is evident that, on any day of the year, the most direct angle of solar radiation at a given location occurs at solar noon. From our previous examples, it follows that for points in the **Northern hemisphere**, the angle of solar elevation at noon is given by:

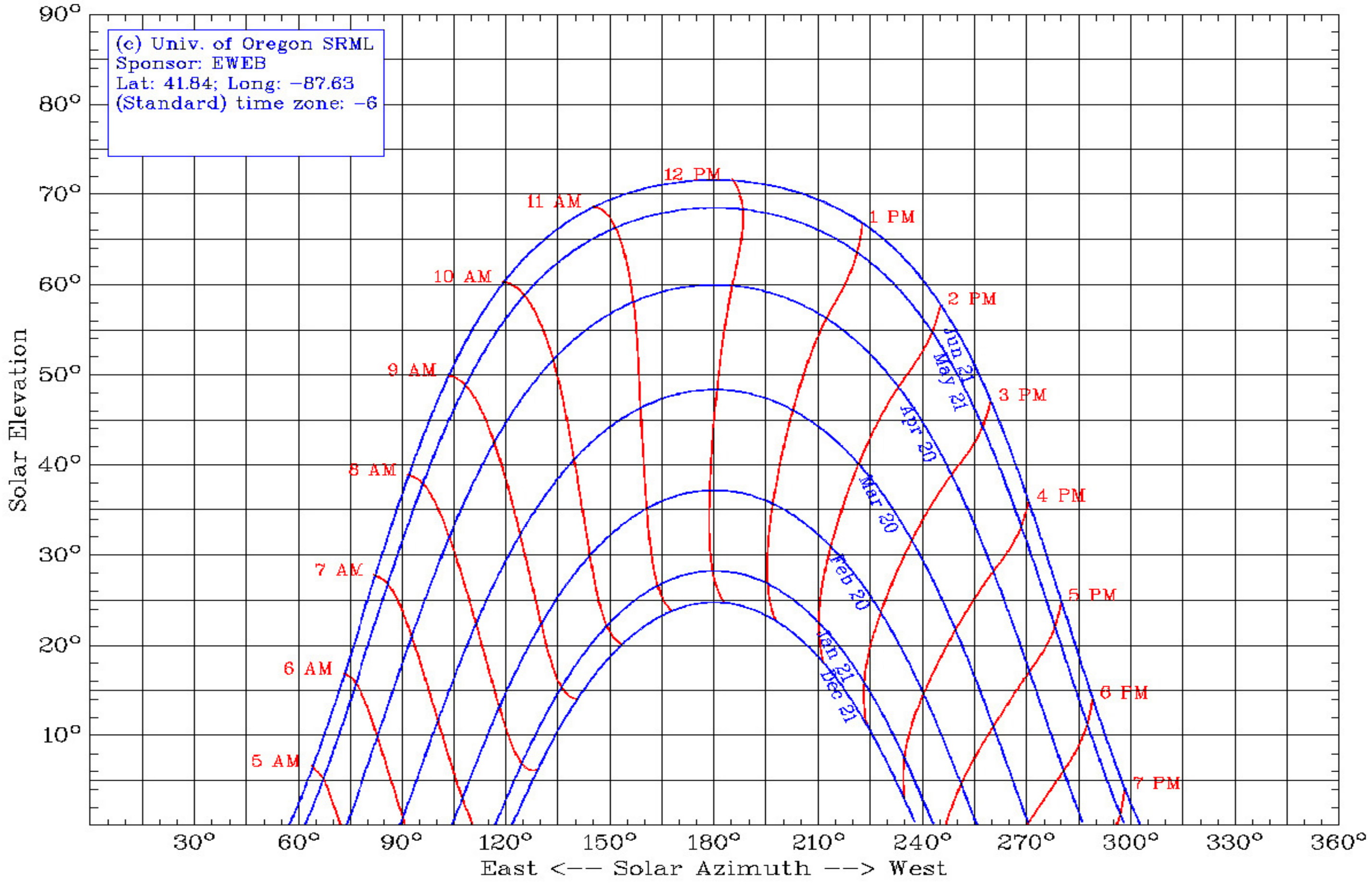
$$\alpha = 90° - (\varphi - \delta) \qquad (2)$$

The symmetrical formula for solar elevation at noon for points in the **Southern hemisphere** is given by:

$$\alpha = 90° + (\varphi - \delta) \qquad (3)$$

We have seen that at solar noon, the range of angles at which the sun strikes the ground at Chicago, Illinois varies from a minimum of 24.7° at the winter solstice to a maximum of 71.7° at the time of the summer solstice.

A **sun path diagram** for Chicago appears below.



In the above diagram, the azimuth and solar elevation angles are the coordinates of an observer's local horizon system in Chicago, Illinois.

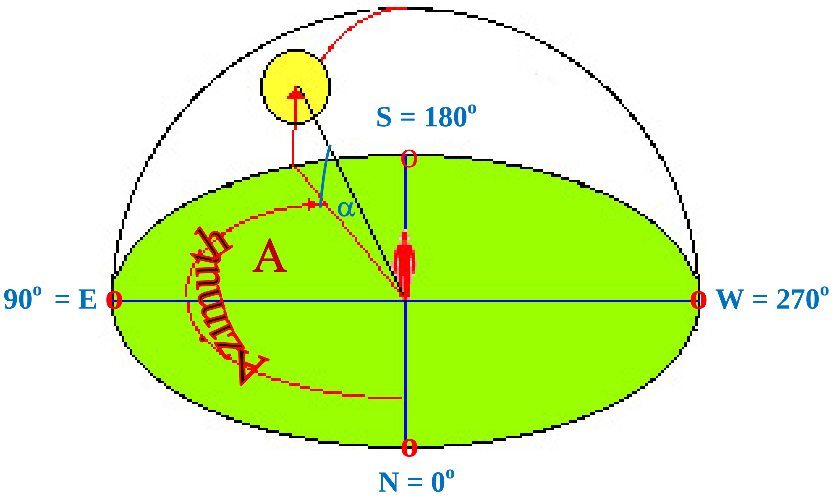
You may obtain a sun path diagram for any specified location on Earth at any specified time of the year by entering the relevant information at the University of Oregon's Solar Radiation Monitoring Laboratory website:

http://solardat.uoregon.edu/SunChartProgram.html .

The **azimuth** is the local angle A between the direction of due North and that of the perpendicular projection of the Sun down onto the horizon line measured clockwise. Thus, we have the azimuth values:

0° = due North, 90° = due East, 180° = due South, and 270° = due West.

The **angle of solar elevation**, α, is defined as before, to be the angular measure of the Sun's rays above the horizon. Equivalently, it is the angle between the direction of the geometric center of the Sun and the horizon.



Sunrise Equation

The **sunrise equation:**

$$\cos \omega_o = -\tan(\varphi)\tan(\delta) \qquad \textbf{(4)}$$

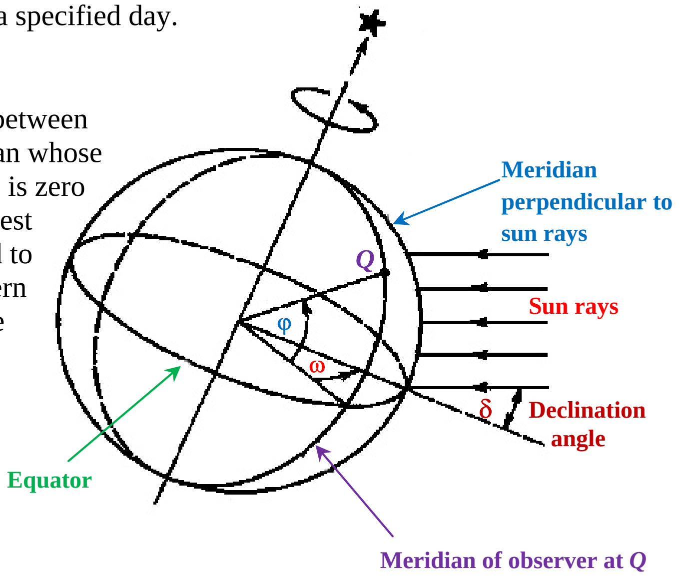
is used to determine the local time of sunrise and sunset at the latitude, φ , when the Sun declination angle is δ degrees. In this equation, we have the quantities:

$\omega = \omega_o$ – the **hour angle** expressed in degrees where
$-180° < \omega_o < 0°$ at sunrise and $0° < \omega_o < +180°$ at sunset.

φ – the latitude of a specified location on Earth

δ – the Sun declination angle on a specified day.

The **hour angle,** ω, is the angular distance between the meridian of the observer and the meridian whose plane contains the sun. Thus, the hour angle is zero at *local noon* (when the sun reaches its highest point in the sky). At this time the sun is said to be 'due south' (or 'due north', in the Southern Hemisphere) since the meridian plane of the observer contains the sun. The hour angle increases by 15 degrees every hour.



It follows that at local noon, the hour angle is zero: $\omega = 0°$. Using a 24 hour time scale for the day, ($0 \le t \le 24$, then $t = 12$ *at local noon*, $t = 6.5$ *at five and one-half hours before local noon* and $t = 18.25$ *at six and one-quarter hours after local noon*) , we have that at time t_o , the measure of the hour angle is:

$$\boxed{\omega = 15 \cdot (t_o - 12)°} \qquad \textbf{(5)}$$

Upon solving (4) for ω_o, setting that result equal to the right side of (5), and then solving it for ($t_o - 12$), we find that

(i) *Sunrise occurs at* $\dfrac{\cos^{-1}(-\tan\varphi \cdot \tan\delta)^o}{15^o}$ hours before local noon

and

(ii) *Sunset occurs at* $\dfrac{\cos^{-1}(-\tan\varphi \cdot \tan\delta)^o}{15^o}$ hours after local noon. **(6)**

Exceptional cases: (a) *If* ($\tan\delta \tan\varphi) \ge 1$, *then no sunset that day*.
(b) *If* ($\tan\delta \tan\varphi) \le -1$, *then no sunrise that day*.

Note that cases (a) and (b) only relate to latitudes beyond ±65.5°, i.e. above the Arctic Circle or below the Antarctic Circle.

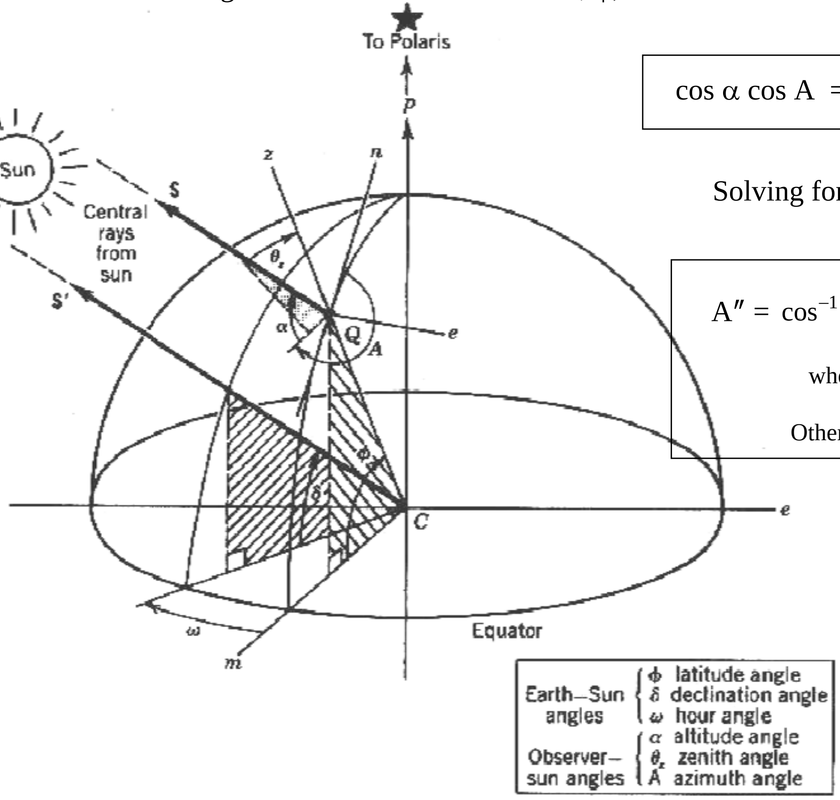
Angle of Solar Elevation for Times of Day Other Than Local Noon

It can be shown that the **angle of solar elevation**, α, may be expressed in terms of the Sun's declination angle, δ, on a given date, the current hour angle (i.e. the time of day) ω, and an observer's latitude, φ, via the equation:

$$\boxed{\alpha = \sin^{-1} (\sin\delta \sin\varphi + \cos\delta \cos\omega \cos\varphi)} \ . \qquad \textbf{(7)}$$

Azimuth Equation

One equation which relates the Sun's azimuth angle, A, at a given location, its angle of elevation, α , the current hour angle ω at the observer's latitude, φ, and the Sun's declination angle, δ, on this date is**:**



$$\boxed{\cos\alpha \cos A = \sin\delta \cos\varphi - \cos\delta \cos\omega \sin\varphi} \qquad \textbf{(8)}$$

Solving for the azimuth, A, we find:

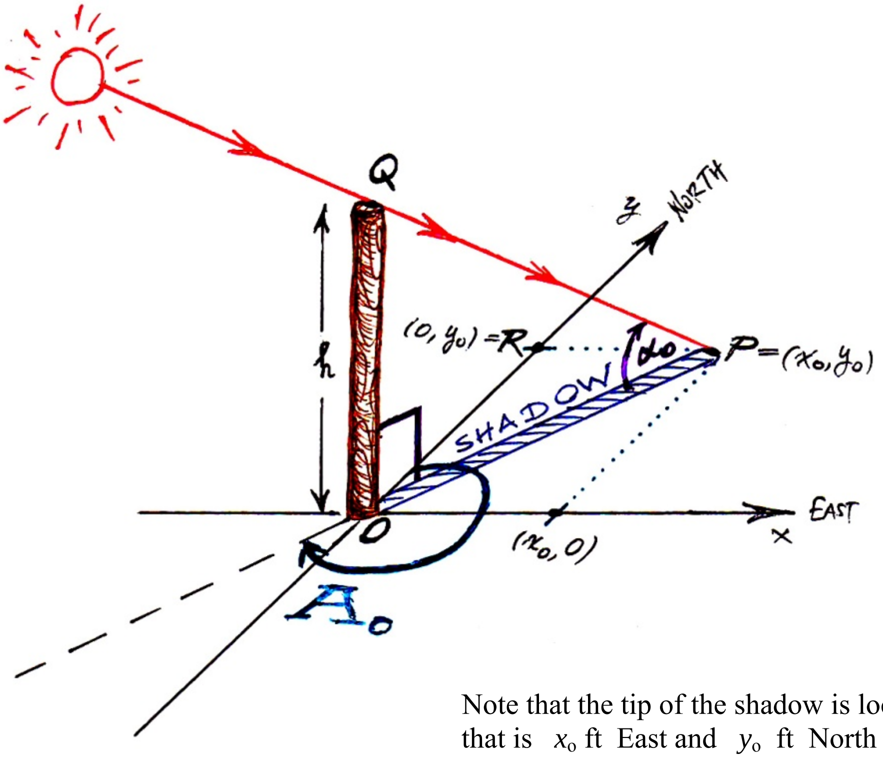
$$A'' = \cos^{-1}\left(\frac{\sin\delta\cos\varphi - \cos\delta\cos\omega\sin\varphi}{\cos\alpha}\right) \qquad \textbf{(9)}$$

where if $\omega \le 0$ then $A = A''$.

Otherwise, $\omega > 0$ and $A = 360° - A''$.

A knowledge of the angle of solar elevation, α, and the azimuth, A, allows us to calculate the length and the location of a simple shadow.

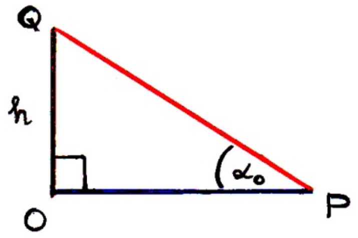
Example Describe the shadow due to a telephone post that is h ft high at the time of day when the solar elevation is $\alpha = \alpha_o$ and the azimuth is $A = A_o$.



In the figure, we observe that this shadow has length $|OP|$ where in the right triangle, $\triangle OPQ$, we find that

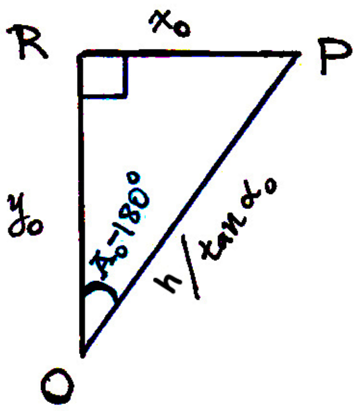
$$\tan \alpha_o = \frac{h}{|OP|}$$

$$\Rightarrow |OP| = \frac{h}{\tan \alpha_o} \text{ ft.}$$



Note that the tip of the shadow is located at the point $P = (x_o, y_o)$ that is x_o ft East and y_o ft North of the base of the post.

By analyzing the right triangle, $\triangle ORP$, we find that



$$\sin(A_o - 180°) = \frac{x_o}{|OP|} \quad \Rightarrow x_o = |OP| \cdot \sin(A_o - 180°) \text{ ft.}$$

Thus, $x_o = \dfrac{h}{\tan \alpha_o} \cdot \sin(A_o - 180°) = -\dfrac{h \sin A_o}{\tan \alpha_o}$ ft,

and

$$\cos(A_o - 180°) = \frac{y_o}{|OP|} \quad \Rightarrow y_o = |OP| \cdot \cos(A_o - 180°) \text{ ft.}$$

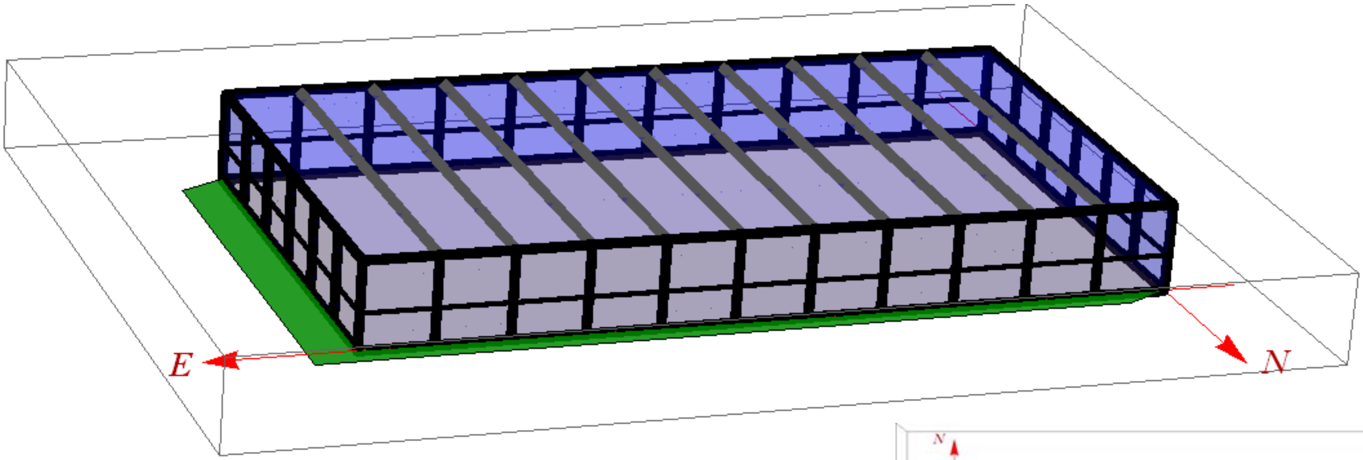
So $y_o = \dfrac{h}{\tan \alpha_o} \cdot \cos(A_o - 180°) = -\dfrac{h \cos A_o}{\tan \alpha_o}$ ft.

Therefore, the shadow's vertex is at

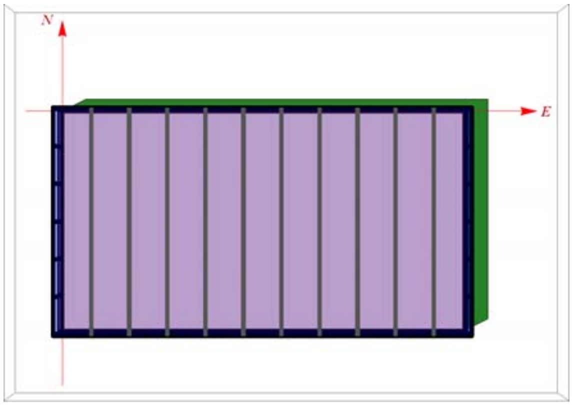
$$P = \left(-\frac{h \sin A_o}{\tan \alpha_o}, -\frac{h \cos A_o}{\tan \alpha_o} \right).$$

Example Describe the shadow positioning due to Crown Hall at various times of day on the occasion of the summer solstice or autumnal equinox.

1. Shadow Position on Summer Solstice 2 Hours After Solar Noon

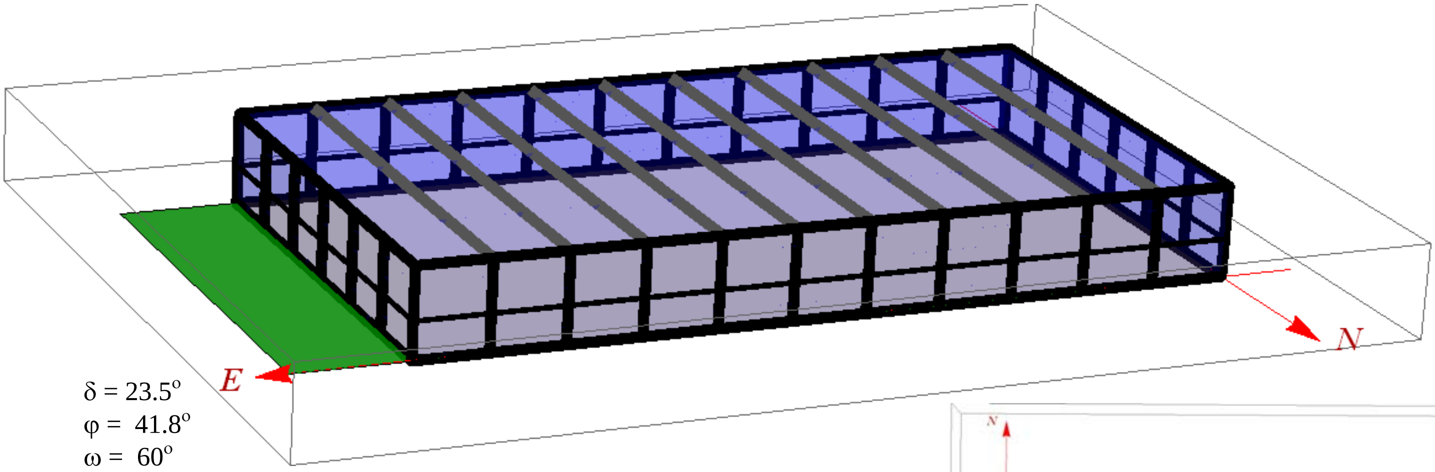


$\delta = 23.5^{o}$
$\varphi = 41.8^{o}$
$\omega = 30^{o}$
$\alpha = 59.1^{o}$
$A = 243.1^{o}$
$A' = A - 180^{o} = 63.1^{o}$
Shadow Length = 14.4 ft
$\Delta y =$ 6.5 ft northward
$\Delta x =$ 12.8 ft eastward

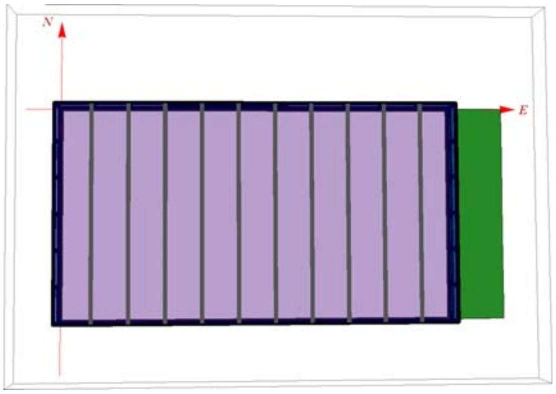


Building Dimensions: 220 ft x 120 ft x 24 ft

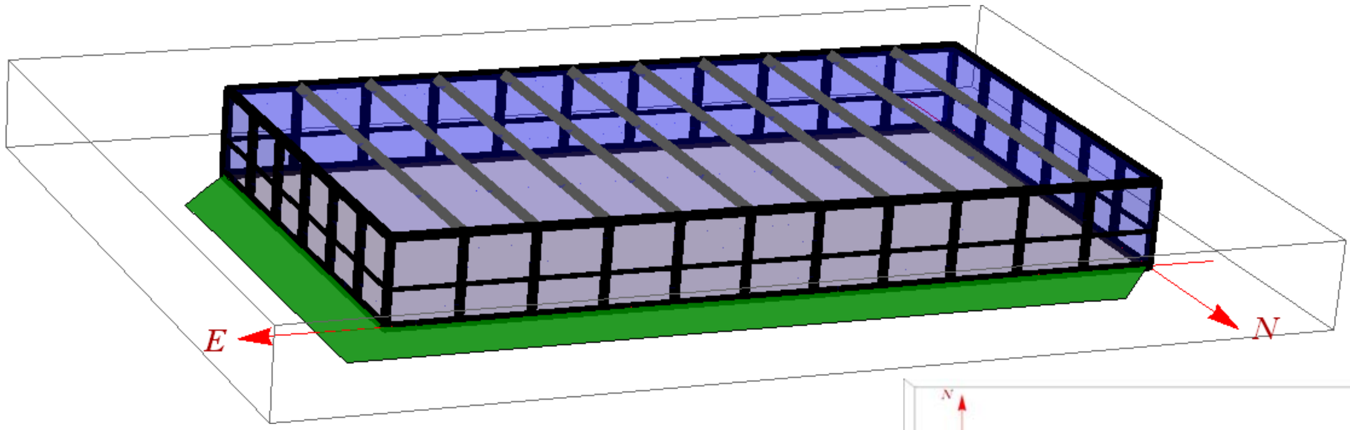
2. Shadow Position on Summer Solstice 4 Hours After Solar Noon

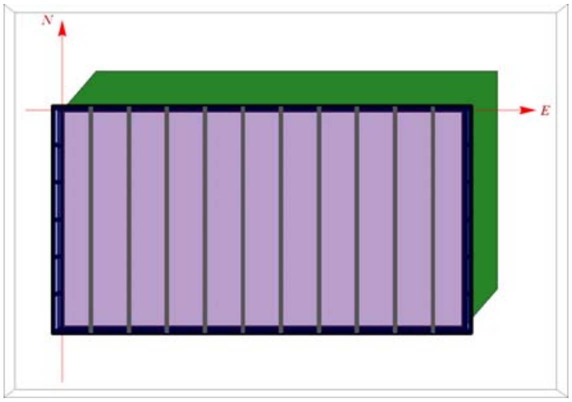


$\delta = 23.5^{o}$
$\varphi = 41.8^{o}$
$\omega = 60^{o}$
$\alpha = 37.4^{o}$
$A = 269.4^{o}$
$A' = A - 180^{o} = 89.4^{o}$
Shadow Length = 31.4 ft
$\Delta y =$ 0.33 ft northward
$\Delta x =$ 31.37 ft eastward



3. Shadow Position on Autumnal Equinox 2 Hours After Solar Noon





$\delta = 0^o$
$\varphi = 41.8^o$
$\omega = 30^o$
$\alpha = 40.2^o$
$A = 220.9^o$
$A' = A - 180^o = 40.9^o$
Shadow Length = 28.4 ft
$\Delta y =$ 21.5 ft northward
$\Delta x =$ 18.6 ft eastward

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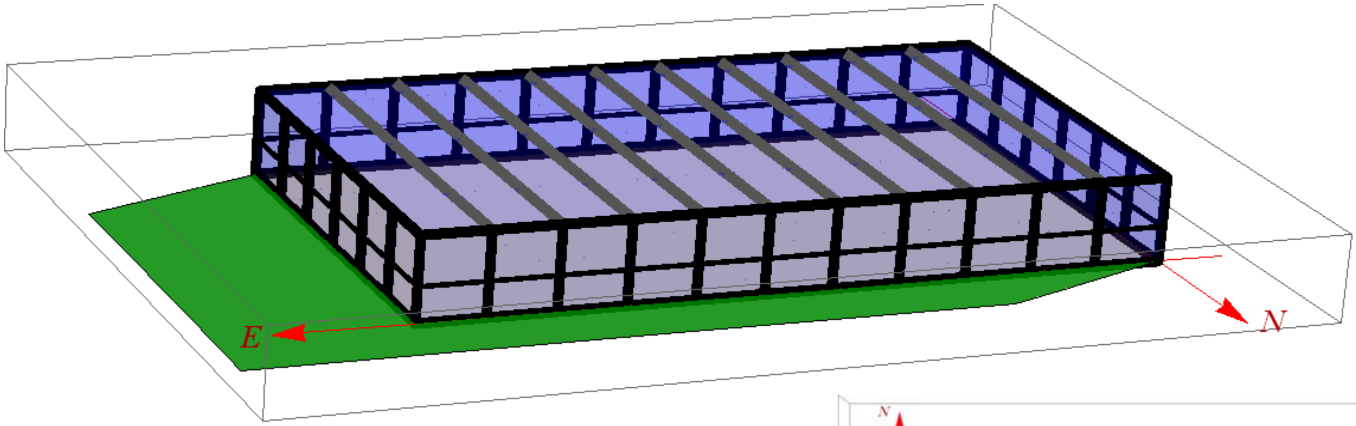
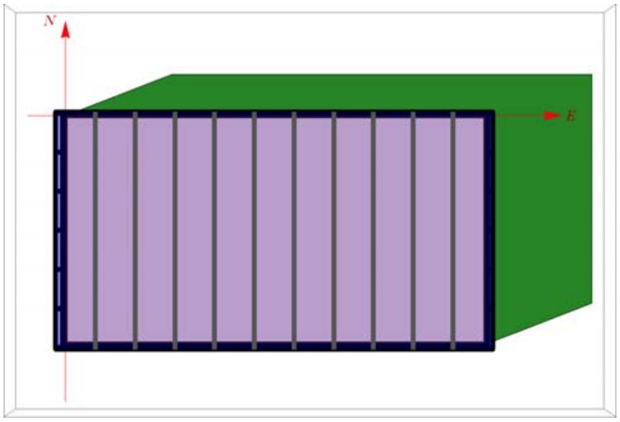
4. Shadow Position on Autumnal Equinox 4 Hours After Solar Noon





$\delta = 0^o$
$\varphi = 41.8^o$
$\omega = 60^o$
$\alpha = 21.9^o$
$A = 249.0^o$
$A' = A - 180^o = 69.0^o$
Shadow Length = 59.7 ft
$\Delta y =$ 21.5 ft northward
$\Delta x =$ 55.8 ft eastward